Lesson 17: Four Interesting Transformations of Functions

Student Outcomes

- Students examine that a vertical translation of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = f(x) + k \).
- Students examine that a vertical scaling of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = kf(x) \).

Lesson Notes

Students enter Algebra 1 having experience with transforming lines, rays, triangles, etc., using translations, rotations, reflections, and dilations from Grade 8 Modules 2 and 3. Thus, it is natural to begin a discussion of transformations of functions by transforming graphs of functions—the graph of a function, \( f: \mathbb{R} \to \mathbb{R} \), is just another geometric figure in the (Cartesian) plane. Students use language such as, “a translation 2 units to the left,” or, “a vertical stretch by a scale factor of 3,” to describe how the original graph of the function is transformed into the new graph geometrically.

As students apply their Grade 8 geometry skills to the graph of the equation \( y = f(x) \), they realize that the translation of the graph to the right by 4 units is given by the graph of the equation \( y = f(x - 4) \). This recognition, in turn, leads to the idea of a transformation of a function. (i.e., a new function such that the graph of it is the transformation of the original graph of \( y = f(x) \).) In the example described, it is the function given by \( g(x) = f(x - 4) \) for any real number \( x \) such that \( x - 4 \) is in the domain of \( f \).

Since the transformation of the function is itself another function (and not a graph), we must use function language to describe the transformation. A function \( f \) cannot be translated up, down, right, or left (even though its graph can). Rather, students can use function language such as: “For the same inputs, the values of the transformed function are two times as large as the values of the original function.”

These lessons encourage fluidity in both the language associated with transformations of graphs and the language associated with transformations of functions. While a formal definition for the transformation of a function is not included, teachers are encouraged use language precisely as students work to develop the notion of transformation of a function and relate it to their understanding of transformations of graphical objects.

In the exploratory challenge, you may highlight MP.3 by asking students to make a conjecture about the effect of \( k \). This challenge also calls on students to employ MP.8, as they will generalize the effect of \( k \) through repeated graphing.
Classwork

Exploratory Challenge 1/Example 1 (12 minutes)

Let $f(x) = |x|$ for all real numbers $x$. Students explore the effect on the graph of $y = f(x)$ by changing the equation $y = f(x)$ to $y = f(x) + k$ for given values of $k$.

**Exploratory Challenge 1/Example 1**

Let $f(x) = |x|, g(x) = f(x) - 3, h(x) = f(x) + 2$ for any real number $x$.

1. Write an explicit formula for $g(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):
   
   $g(x) = |x| - 3$

2. Write an explicit formula for $h(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):
   
   $h(x) = |x| + 2$

3. Complete the table of values for these functions.

| $x$ | $f(x) = |x|$ | $g(x) = f(x) - 3$ | $h(x) = f(x) + 2$ |
|-----|-------------|-----------------|-----------------|
| −3  | 3           | 0               | 5               |
| −2  | 2           | −1              | 4               |
| −1  | 1           | −2              | 3               |
| 0   | 0           | −3              | 2               |
| 1   | 1           | −2              | 3               |
| 2   | 2           | −1              | 4               |
| 3   | 3           | 0               | 5               |

4. Graph all three equations: $y = f(x), y = f(x) - 3,$ and $y = f(x) + 2$. 

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Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.
5. What is the relationship between the graph of \( y = f(x) \) and the graph of \( y = f(x) + k \)?

For values of \( k \), where \( k > 0 \), for every point \((x, f(x))\) that satisfies the equation \( y = f(x) \), there is a corresponding point \((x, f(x) + k)\) on the graph, located \( k \) units above \((x, f(x))\) that satisfies the equation \( y = f(x) + k \). The graph of \( y = f(x) + k \) is the vertical translation of the graph of \( y = f(x) \) by \( k \) units upward.

For values of \( k \), where \( k < 0 \), for every point \((x, f(x))\) that satisfies the equation \( y = f(x) \), there is a corresponding point \((x, f(x) + k)\) on the graph, located \( k \) units below \((x, f(x))\) that satisfies the equation \( y = f(x) + k \). The graph of \( y = f(x) + k \) is the vertical translation of the graph of \( y = f(x) \) by \( k \) units downward.

The use of transformation language like “vertical translation” is purposeful. The Common Core State Standards require students to spend a lot of time talking about translations, rotations, reflections, and dilations in Grade 8. They will spend more time in Grade 10. Reinforcing this vocabulary will help to link these grades together.

6. How do the values of \( g \) and \( h \) relate to the values of \( f \)?

For each \( x \) in the domain of \( f \) and \( g \), the value of \( g(x) \) is 3 less than the value of \( f(x) \). For each \( x \) in the domain of \( f \) and \( h \), the value of \( h(x) \) is 2 more than the value of \( f(x) \).

Discussion (3 minutes)

Students should finish Example 1 with the understanding that the graph of a function \( g \) found by adding a number to another function, as in \( g(x) = f(x) + k \), is the translation of the graph of the function \( f \) vertically by \( k \) units (positively or negatively depending on the sign of \( k \)).

**Exploratory Challenge 2/Example 2 (12 minutes)**

Let \( f(x) = |x| \) for any real number \( x \). Students explore the effect on the graph of \( y = f(x) \) by changing the equation \( y = f(x) \) to \( y = kf(x) \) for given values of \( k \).

**Exploratory Challenge 2/Example 2**

1. Let \( f(x) = |x| \), \( g(x) = 2f(x) \), \( h(x) = \frac{1}{2}f(x) \) for any real number \( x \). Write a formula for \( g(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):

\[
g(x) = 2|x|
\]

2. Write a formula for \( h(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):

\[
h(x) = \frac{1}{2}|x|
\]
3. Complete the table of values for these functions.

| x   | \( f(x) = |x| \) | \( g(x) = 2f(x) \) | \( h(x) = \frac{1}{2}f(x) \) |
|-----|-----------------|-----------------|-----------------|
| −3  | 3               | 6               | 1.5             |
| −2  | 2               | 4               | 1               |
| −1  | 1               | 2               | 0.5             |
| 0   | 0               | 0               | 0               |
| 1   | 1               | 2               | 0.5             |
| 2   | 2               | 4               | 1               |
| 3   | 3               | 6               | 1.5             |

4. Graph all three equations: \( y = f(x) \), \( y = 2f(x) \), and \( y = \frac{1}{2}f(x) \).

Given \( f(x) = |x| \), let \( p(x) = -|x| \), \( q(x) = -2f(x) \), \( r(x) = -\frac{1}{2}f(x) \) for any real number \( x \).

5. Write the formula for \( q(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[ q(x) = -2|x| \]

6. Write the formula for \( r(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[ r(x) = -\frac{1}{2}|x| \]
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7. Complete the table of values for the functions \( p(x) = -|x|, q(x) = -2f(x), r(x) = -\frac{1}{2}f(x) \).

| \( x \) | \( p(x) = -|x| \) | \( q(x) = -2f(x) \) | \( r(x) = -\frac{1}{2}f(x) \) |
|------|-------------|-------------|-------------|
| -3   | -3          | -6          | -1.5        |
| -2   | -2          | -4          | -1          |
| -1   | -1          | -2          | -0.5        |
| 0    | 0           | 0           | 0           |
| 1    | -1          | -2          | 0.5         |
| 2    | -2          | -4          | -1          |
| 3    | -3          | -6          | -1.5        |

8. Graph all three functions on the same graph that was created in Problem 4. Label the graphs as \( y = p(x), y = q(x), \) and \( y = r(x). \)

9. How is the graph of \( y = f(x) \) related to the graph of \( y = kf(x) \) when \( k > 1 \)?

The graph of \( y = kf(x) \) for \( k > 1 \) contains points \((x, k\cdot y)\) which are related to points \((x, y)\) in the graph of \( y = f(x)\). The number \( k \) is a multiple of \( y\): each \( y\)-value of \( y = g(x) \) is \( k \) times the \( y\)-value of \( y = f(x) \). The graph of \( y = kf(x) \) is a vertical scaling that appears to stretch the graph of \( y = f(x) \) vertically by a factor of \( k \).
10. How is the graph of \( y = f(x) \) related to the graph of \( y = kf(x) \) when \( 0 < k < 1 \)?

The graph of \( y = kf(x) \) for \( 0 < k < 1 \) contains points \((x, ky)\) which are related to points \((x, y)\) in the graph of \( y = f(x) \). The number \( ky \) is a fraction of \( y \): each \( y \)-value of \( y = g(x) \) is \( k \) times the \( y \)-value of \( y = f(x) \). The graph of \( y = kf(x) \) is a vertical scaling that appears to shrink the graph of \( y = f(x) \) vertically by a factor of \( k \).

11. How do the values of functions \( p, q, \) and \( r \) relate to the values of functions \( f, g, \) and \( h \), respectively? What transformation of the graphs of \( f, g, \) and \( h \) represents this relationship?

Each function is the opposite of the corresponding function. The result is that each \( y \)-value of any point on the graph of \( y = p(x) \), \( y = q(x) \), and \( y = r(x) \) are the opposite of the \( y \)-value of the graphs of the equations \( y = f(x), y = g(x) \), and \( y = h(x) \). Each graph is a reflection of the corresponding graph over the \( x \)-axis.

Discussion (3 minutes)

Students should finish Example 2 with the understanding that a number, a scale factor, multiplied to a function vertically scales the original graph. For a vertical scale factor of \( k > 1 \), the graph is a vertical stretch of the original graph; for a vertical scale factor of \( k \) where \( 0 < k < 1 \), the graph is a vertical shrink of the original graph. For a vertical scale factor of \( k \) where \( -1 < k < 0 \), the graph of the function is a reflection across the \( x \)-axis of the graph when \( 0 < k < 1 \). Similarly, for a vertical scale factor of \( k < -1 \), the graph is the reflection across the \( x \)-axis of the graph when \( k > 1 \).

Exercises (8 minutes)

Students complete exercises independently; then compare/discuss with partner or small group. Circulate to ensure that students grasp the effects of the given transformations.

**Exercises**

1. Make up your own function \( f \) by drawing the graph of it on the Cartesian plane below. Label it as the graph of the equation, \( y = f(x) \). If \( b(x) = f(x) - 4 \) and \( c(x) = \frac{1}{4} f(x) \) for every real number \( x \), graph the equations \( y = b(x) \) and \( y = c(x) \) on the same Cartesian plane.

   *Answers will vary. Look for and encourage students to create interesting graphs for their function \( f \). (Functions DO NOT have to be defined by algebraic expressions—any graph that satisfies the definition of a function will do.) One such option is using \( f(x) = |x| \), as shown in the example below."
If time permits, have students present their graphs to the class and explain how they found the graphs of \( y = b(x) \) and \( y = c(x) \). Pay close attention to how students explain how they found the graph of \( y = c(x) \). Many might actually describe a horizontal scaling (or some other transformation that takes each point \((x, y)\) of the graph to another point that does not have the same \(x\)-coordinate). Stress that multiplying the function \( f \) by \( k \) only scales the \(y\)-coordinate and leaves the \(x\)-coordinate alone.

**Closing (3 minutes)**

Point out that there is nothing special about using the function \( f(x) = |x| \) as we did in this lesson. These transformations hold in general:

- Discuss how the graph of \( y = f(x) \) can be vertically translated by positive or negative \( k \). Draw a graph of a made up function on the board, labeled by \( y = f(x) \), and show how to translate it up or down by \( k \) using the equation \( y = f(x) + k \).
- Discuss how the graph of \( y = f(x) \) can be vertically scaled by \( k \) for \( 0 < k < 1, k > 1, -1 < k < 0, k < -1 \). Use the graph of \( y = f(x) \) to show how to vertically scale (i.e., vertically stretch or shrink) by \( k \) units using the equation \( y = kf(x) \).

**Exit Ticket (5 minutes)**
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Exit Ticket

Let \( p(x) = |x| \) for every real number \( x \). The graph of \( y = p(x) \) is shown below.

1. Let \( q(x) = -\frac{1}{2}|x| \) for every real number \( x \). Describe how to obtain the graph of \( y = q(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = q(x) \) on the same set of axes as the graph of \( y = p(x) \).

2. Let \( r(x) = |x| - 1 \) for every real number \( x \). Describe how to obtain the graph of \( y = r(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = r(x) \) on the same set of axes as the graphs of \( y = p(x) \) and \( y = q(x) \).
Exit Ticket Sample Solutions

Let \( p(x) = |x| \) for every real number \( x \). The graph of \( y = p(x) \) is shown below.

1. Let \( q(x) = -\frac{1}{2}|x| \) for every real number \( x \). Describe how to obtain the graph of \( y = q(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = q(x) \) on the same set of axes as the graph of \( y = p(x) \).

   Reflect and vertically scale the graph of \( y = p(x) \) by plotting \( (x, -\frac{1}{2}y) \) for each point \((x, y)\) in the graph of \( y = p(x) \). See the graph of \( q(x) \) below.

2. Let \( r(x) = |x| - 1 \) for every real number \( x \). Describe how to obtain the graph of \( y = r(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = r(x) \) on the same set of axes as the graphs of \( y = p(x) \) and \( y = q(x) \).

   Translate the graph of \( y = p(x) \) vertically down 1 unit. See the graph of \( y = r(x) \) below.

Problem Set Sample Solutions

Let \( f(x) = |x| \) for every real number \( x \). The graph of \( y = f(x) \) is shown below. Describe how the graph for each function below is a transformation of the graph of \( y = f(x) \). Then use this same set of axes to graph each function for problems 1 - 5. Be sure to label each function on your graph (by \( y = a(x) \), \( y = b(x) \), etc.).

1. \( a(x) = |x| + \frac{3}{2} \)

   Translate the graph of \( y = f(x) \) up 1.5 units.

2. \( b(x) = -|x| \)

   Reflect \( y = f(x) \) across the \( x \)-axis.
3. \( c(x) = 2|x| \\
   \text{Vertically scale/stretch the graph of } y = f(x) \text{ by doubling the output values for every input.}

4. \( d(x) = \frac{1}{3}|x| \\
   \text{Vertically scale/shrink the graph of } y = f(x) \text{ by dividing the output values by 3 for every input.}

5. \( e(x) = |x| - 3 \\
   \text{Translate the graph of } y = f(x) \text{ down 3 units.}

6. Let \( r(x) = |x| \) and \( t(x) = -2|x| + 1 \) for every real number \( x \). The graph of \( y = r(x) \) is shown below. Complete the table below to generate output values for the function \( t \); then graph the equation \( y = t(x) \) on the same set of axes as the graph of \( y = r(x) \).

| \( x \) | \( r(x) = |x| \) | \( t(x) = -2|x| + 1 \) |
|---|---|---|
| -2 | 2 | -3 |
| -1 | 1 | -1 |
| 0 | 0 | 1 |
| 1 | 1 | -1 |
| 2 | 2 | -3 |
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Let $f(x) = |x|$ for every real number $x$. Let $m$ and $n$ be functions found by transforming the graph of $y = f(x)$. Use the graphs of $y = f(x)$, $y = m(x)$, and $y = n(x)$ below to write the functions $m$ and $n$ in terms of the function $f$. (Hint: what is the $k$?)

- $m(x) = 2f(x)$
- $n(x) = f(x) + 2$